Discrete Mathematics - 2 Marks.

Unit-I Logic and Proofs

1. Get the contra positive of the statement "If it is raining the I get wet".

Ans: Let p: it is raining

q: I get we

Given $p \to q$. Its contrapositive is given by $\neg q \to \neg p$

(ie) If I don't get wet then it is not raining.

2. Is it true that the negation of a conditional statement is also a conditional statement?

Ans: No, because $\neg(p \to q) = \neg(\neg p \vee q) = p \wedge \neg q$

3. Find a counter example, if possible, to these universelly quantified statements, whose the universe of discourse for all variables consists of all integers.

(a) $\forall x \forall y (x^2 = y^2 \to x = y)$. (b) $\forall x \forall y (xy \geq x)$

Ans: (a) $x = 2$, $y = -2$

(b) $17 = x$, $y = -1$

4. Show that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$ without using truth table.

Ans: $p \to (q \to r) \Leftrightarrow \neg p \vee (\neg q \vee r)$

$\Leftrightarrow (\neg p \vee \neg q) \vee r$

$\Leftrightarrow \neg(p \wedge q) \vee r$

$\Leftrightarrow (p \wedge q) \to r$.

5. Show that $(\neg p) \to (p \to q)$ is a tautology.

Ans: $(\neg p) \to (p \to q) \Leftrightarrow p \vee (\neg p \vee q)$

$\Leftrightarrow (p \vee \neg p) \vee q$

$\Leftrightarrow T \vee q$

$\Leftrightarrow T$.

6. Write the truth table for the formula $(p \wedge q) \vee (\neg p \wedge \neg q)$

<u>Ans:</u>

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

7. Express in symbolic form, everyone who is healthy can do all kinds of work.

<u>Ans:</u> Let $P(x)$: x is healthy and $Q(x)$: x do all work.

Symbolic form $\forall x\,(P(x) \rightarrow Q(x))$

8. Write the negation of the statement "If there is a will, then there is a way"

<u>Ans:</u>

Let p : There is a will

q : There is a way.

Given $p \rightarrow q \Leftrightarrow \neg p \vee q$

Its negation is given by "There is a will and there is no way".

9. When do yo say that two compound propositions are equivalent?

<u>Ans:</u> Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ is a tautology. It is denoted by the symbol $A \Leftrightarrow B$ which is read as "A is equivalence to B".

10. Prove that $(p \leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

<u>Ans:</u> $(p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

$\Leftrightarrow (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (p \wedge q)$

$\Leftrightarrow (\neg p \wedge \neg q) \vee (p \wedge q)$

11. Write the statement in symbolic form "Some real numbers are rational".

<u>Ans:</u> Let $R(x)$: x is a real number ; $Q(x)$: x is rational.

Symbolic form: $\exists x\,(R(x) \wedge Q(x))$.

12. Rewrite the following using quantifiers "Every student in the class studied calculus?

Ans:

Let $P(x)$: x is a student.

$Q(x)$: x studied calculus.

Symbolic form : $\forall x (P(x) \rightarrow Q(x))$.

13. Check whether $((p \rightarrow q) \rightarrow r) \vee \neg p$ is a tautology.

Ans:

$$((p \rightarrow q) \rightarrow r) \vee \neg p \Leftrightarrow ((\neg p \vee q) \rightarrow r) \vee \neg p.$$

$$\Leftrightarrow (\neg (\neg p \vee q) \vee r) \vee \neg p$$

$$\Leftrightarrow (p \wedge \neg q) \vee (r \vee \neg p)$$

$$\Leftrightarrow (r \vee \neg p \vee p) \wedge (r \vee \neg p \vee \neg q)$$

$$\Leftrightarrow T \wedge (r \vee \neg p \vee \neg q)$$

$$\Leftrightarrow (r \vee \neg p \vee \neg q)$$

$\therefore$ The given statement is not a tautology.

14. Define compound statement formula.

Ans: An expression consisting of simple statement functions (one or more variables) connected by logical connectives are called a compound statement.

15. Write the statement in the symbolic form, " Some integers are not square of any integers ".

Ans: Let $I(x)$: x is an integer.

$S(x)$: x is a square of any integer.

$*$ Symbolic form: $\exists x (I(x) \wedge \neg S(x))$

16. Define contradiction:

Ans: A propositional formula which is always false irrespective of the truth values of the individual variables is a contradiction.

Unit-II Combinatorics

1. State pigeo hole principle.

   Ans: If $(n+1)$ pigeon occupies n holes then atleast one hole hasmore than 1 pigeon.

2. State the generalized pigeon hole principle.

   Ans: If m pigeons occupies n holes $(m>n)$, thenat least onehole has more that $\left\lfloor\frac{m-1}{n}\right\rfloor+1$ pigeons.

3. In how many ways can 6 persons occupy 3 vacant seats?

   Ans: Total no of ways = $^6C_3 = 20$ ways.

4. Show that, among 100 people, at least 9 of them were born in the same month.

   Ans: Here no. of pigeon = m = no. of people $=100$

   No. of holes = n = no. of months $=12$

   Then by generalized pigeon hole principle, $\left\lfloor\frac{100-1}{12}\right\rfloor+1 = 8+1 = 9$ were born in the same month.

5. How many permutations of the letters in ABCDEFGH contain the string ABC?

   Ans: Because the letters ABC must occure as block, we can find the answer by finding no. of permutation of six objects, namely the block ABC and individual letters D, E, F, G and H. Therefore, there are $6! = 720$ permutations of the letters in ABCDEFGH which Contains the string ABC.

6. How many different bit strings are there of length 7?

   Ans: By product rule, $2^7 = 128$ ways.

7. If the sequence $a_n = 3\cdot 2^n$, $n \geq 1$, then find the corresponding reccurrence relation.

   Ans: For $n\geq 1$, $a_n = 3\cdot 2^n$, $a_{n-1} = 3\cdot 2^{n-1} = 3\cdot\frac{2^n}{2} \Rightarrow 2a_{n-1} = a_n$

$\therefore a_n = 2a_{n-1}$ for $n \geq 1$ with $a_0 = 3$.

8. If seven colurs are use to paint 50 bicycles, then show that at least 8 bicycles will be same colour.

Ans: Here. No. of pigeon = m = No. of bicycle = 50

No of Holes = n = No. of colours = 7.

By generalized pigeon hole principle, we have $\left[\frac{50-1}{7}\right]+1 = 8$

9. Find the reccurrence relation whose solution is $s(k) = 5 \cdot 2^k$

Ans: Given $s(k) = 5 \cdot 2^k$

$\Rightarrow s(k-1) = 5 \cdot 2^{k-1}$

$= \frac{5}{2} \cdot 2^k$

$\Rightarrow 2 s(k-1) = s(k)$

$\Rightarrow 2\, s(k-1) - s(k) = 0$ with $s(0) = 5$ is the required reccurrence relation.

10. Find the associated homogeous solution for $a_n = 3a_{n-1} + 2n$.

Ans: Its associated homogeous equation is $a_n - 3a_{n-1} = 0$.

Its characteristic equation is $r - 3 = 0 \Rightarrow r = 3$.

Now, the solution of associated homogeneous equation is $a_n = A \cdot 3^n$.

11. Define Generating function.

Ans: The generating function for the sequence 's' with terms $a_0, a_1, \ldots a_n, \ldots$ of real numbers is the infinite sum.

$G(x) = G(s, x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$.

12. Find the generating function for the sequence 's' with terms 1, 2, 3, 4 ......

Ans: $G(x) = G(s, x) = \sum_{n=0}^{\infty} (n+1)\, x^n$

$= 1 + 2x + 3x^2 + \cdots$

$= (1-x)^{-2} = \frac{1}{(1-x)^2}$

13. How many permutations of (a, b, c, d, e, f, g) and with a?

Ans: $6! \times 1! = 720$.

14. Find the number of arrangements of the letters in MAPPANASSRR.

Ans:

Number of arrangements $= \frac{11!}{3!\,2!\,2!} = \frac{3991680}{48}$

15. In how many ways can letters of the word "INDIA" be arranges?

Ans: The word contains 5 letters of which 2 are I's

The number of words possible $= \frac{5!}{2!} = 60$.

16. How many students must be in a class to guarantee that atleast two students recieve the same score on the final exam of the exam if the exam is graded on a scale from 0 to 100 points.

Ans: There are 101 possible scores as 0, 1, 2, ... 100.

By Pigeon hole principle, we have among 102 students there must

be atleast two students with the same score.

The class should contain minimum 102 students.

17. Show that among any group of five (not necessarily consecutive) integers, there are two with same remainder when divided by 4.

Soln:

Take any group of five integers. When these are divided by 4 each have some remainder. Since there are five integers and four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, atleast two have the same remainder.

Unit - III Graph Theory

1. Define Graph:

Ans: A graph $G = (V, E)$ consists of a finite non-empty set V, the element of which are the vertices of G, and a finite set E of unordered pairs of distinct elements of V called the edges of G.

2. Define complete graph.

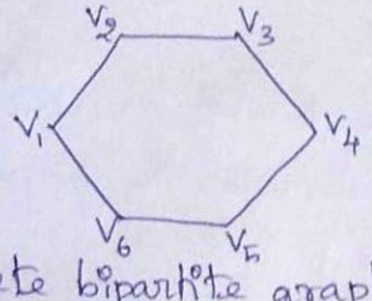
Ans: A graph of n vertices having each pair of distinct vertices joined by an edge is called a Complete graph and is denoted by K_n.

3. Define regular graph:

Ans: A graph in which each vertex has the same degree is called a regular graph. A regular graph has k-regular if each vertex has degree k.

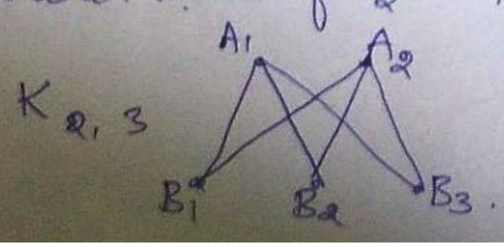
4. Define Bipartite graph with example.

Ans: Let $G = (V, E)$ be a graph. G is bipartite graph if its vertex set V can be partitioned into two non-empty disjoint subsets V_1 and V_2, called a bipartition, such that each edge has one end in V_1 and in V_2. For example, C_6.



5. Define complete bipartite graph with example.

Ans: A complete bipartite graph is a bipartite graph with bipartition V_1 and V_2 in which each vertex of V_1 is joined by an edge to each vertex of V_2. For example,



6. Define subgraph.

Ans: A graph $H = (V_1, E_1)$ is a subgraph of $G = (V, E)$ provided that $V_1 \subseteq V$, E_1 and for each $e \subset E_1$, both ends of e are in V_1.

7. Define Isomorphism of two graphs.

Ans: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the same or isomorphic, if there is a bijection $F : V_1 \to V_2$ such that $(u, v) \in E_1$ iff $(F(u), F(v)) \in E_2$.

8. Define strongly connected graph.

Ans: A digraph G is said to be strongly connected if for every pair of vertices, both vertices of the pair are reachable from one another.

9. State the necessary and sufficient conditions for the existance of an Eulerian path in connected graph.

Ans: A connected graph contains an Euler path iff it has exactly two vertices of odd degree.

10. State Handshaking theorem.

Ans: If $G = (V, E)$ is an undirected graph with e edges, then

$$\sum_i \deg(v_i) = 2e.$$

11. Define adjacency matrix

Ans: Let $G = (V, E)$ be a graph with n vertices.

An $n \times n$ matrix A is an adjacency matrix for G iff for $i \le I$, $j \le n$, $A(i,j) = \begin{cases} 1, & \text{for } (i,j) \text{ in } E \\ 0, & \text{for } (i,j) \text{ is not in } E. \end{cases}$

12. Define connected graph.

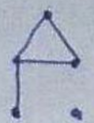
Ans: A graph for which each pair of vertices is joined by a trail is connected.

13. Define Pseudo-graph.

Ans: A graph is called a pseudo-graph if it has both parallel edges and self loobs.

14. Does there exist a simple graph with five vertices of the 0, 1, 2, 2, 3 degrees? If so, draw such a graph.

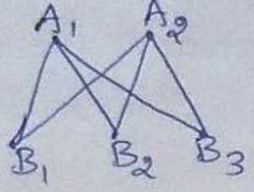
Ans: Yes.



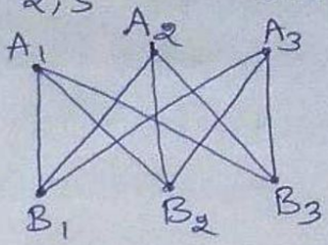
15. Draw a complete bipartite graph of $K_{2,3}$ and $K_{3,3}$

Ans:

$K_{2,3}$



$K_{3,3}$



16. Define spanning subgraph.

Ans: Let a graph $H = (V_1, E_1)$ is a subgraph of $G = (V, E)$

H is a spanning subgraph of G if H is a subgraph of G with both ~~and~~ $V_1 = V$ and $E_1 \subset E$.

17. Define Induce subgraph.

Ans: A graph $H = (V_1, E_1)$ is a subgraph of $G = (V, E)$.

H is an induced subgraphs of G such that E_1 consists of all the edges of G with both ends in V_1.

18. Define Eulerian circuit.

Ans: A. circuit in a graph that includes each edge exactly once, the circuit is called an Euclidian circuit.

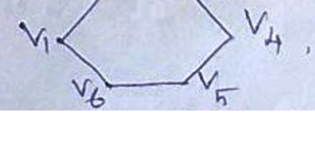
19. State the condition ~~of~~ for Eulerian cycle.

Ans: (i) Starting and ending points are same.

(ii) Cycle should contain all edges of graph but exactly once.

20. Show that C_6 is a bipartite graph?

Ans: C_6 vertex set is partitioned into two set $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, where every edge of C_6 joins a vertex in V_1 to a vertex V.



Unit - IV Group Theory.

1. Define Algebraic System

Ans: A set together with one or more n-ary operations on it is called an algebraic system.

Example: $(\mathbb{Z}, +)$ is an algebraic system.

2. Define Semi group.

Ans: Let S be non-empty set, $*$ be a binary operation on S. The algebraic system $(S, *)$ is called a semigroup, if the operation is associative. In otherwords $(S, *)$ is a semi group if for any $x, y, z \in S$, $x * (y * z) = (x * y) * z$.

3. Define Monoid:

Ans: A semi group $(M, *)$ with identity element with respect to the operation $*$ is called a Monoid.

4. Define Group.

Ans: An algebraic system $(G, *)$ is called a group if it satisfies the following properties.

(i) G is closed with respect to $*$

(ii) $*$ is associative

(iii) identity element exists

(iv) inverse element exists.

5. State any two properties of a group.

Ans: (i) The identity element of a group is unique.

(ii) The inverse of each element is unique.

6. Define Commutative ring.

Ans: If ~~the ring~~ $(R, *)$ is commutative, then the ring $(R, +, *)$ is called a commutative ring.

7. Show that the inverse of an element in a group $(G, *)$ is unique.

Ans: Let $(G, *)$ be a group with identity element e.

Let 'b' and 'c' be inverse of an element 'a'.

(ie) $a * b = b * a = e$ and $a * c = c * a = e$

Now, $b = b * e = b * (a * c) = (b * a) * c = e * c = c$

$\therefore b = c$

Hence inverse element is unique.

8. Give an example of semigroup but not a Monoid.

Ans: The set of all positive integers over addition from a semi-group but it is not a Monoid.

9. Prove that the semigroup homomorphism preserves idempotency

Ans: Let $a \in S$ be an idempotent element.

$\therefore a * a = a$.

$\Rightarrow g(a*a) = g(a)$ and

$g(a) \circ g(a) = g(a)$

This shows that $g(a)$ is an idempotent element in T.

Therefore the property of idempotency is preserved under semigroup homomo-rphism.

10. Define cyclic group

Ans: A group $(G,*)$ is said to be cyclic if there exists an element $a \in G$ such that every element of G can be written as some power of 'a'.

11. Define group homomorphism.

Ans: Let $(G,*)$ and $(S,\circ)$ be two groups. A mapping $f: G \to S$ is said to be a group homomorphism if for any $a,b \in G$, $f(a*b) = f(a) \circ f(b)$

12. Define Left coset.

Ans: Let $(H,*)$ be a subgroup of $(G,*)$. For any $a \in G$, the set H is defined by $aH = \{a*h : h \in H\}$ is called the left coset of H determined by $a \in G$.

13. State Lagrange's theorem

Ans: The order of the subgroup of a finite group G divides the order of the group.

14. Define Ring.

Ans: An algebraic system $(R,+,*)$ is called a ring if the binary operations + and * satisfies the following.

(i) $(R,+)$ is an abelian group (ii) $(R,*)$ is a semi group

(iii) the operation + is distributive over *.

15. Define Integral Domain

Ans: A commutative ring R with a unit element is called an integral domain if R has no zero divisors.

16. Show that the semi group homomorphism preserves the property of idempotency.

Ans: Let $f:(M,*) \to (H,\Delta)$ be a semi group homomorphism. x is an idempotent element in M.

$x * x = x$ $\therefore f(x*x) = f(x) \Delta f(x)$.

Unit-V Lattices and Boolean Algebra

1. Define Lattice

Ans: A partially ordered set $(L, \leq)$ in which every pair of elements has a least upper bound and greatest lower bound is called a lattice.

2. Define Lattice homomorphism and isomorphism

Ans: If $(L_1, \wedge, \vee)$ and $(L_2, \oplus, *)$ are two lattices, a mapping $f: L_1 \to L_2$ is called a lattice homomorphism from L_1 to L_2, if for any $a, b \in L_1$, $f(a \vee b) = f(a) \oplus f(b)$ and $f(a \wedge b) = f(a) * f(b)$.

If a homomorphism $f: L_1 \to L_2$ of two lattices $(L_1, \wedge, \vee)$ and $(L_2, \oplus, *)$ is objective. (ie) one-one, onto, then f is called an isomorphism.

3. Define sub lattice with example.

Ans: A non-empty subset M of a lattice $(L, \wedge, \vee)$ is called a sub lattice of L, iff M is closed under both the operations $\wedge$ and $\vee$ that is if $a, b \in M$, then $a \vee b$ and $a \wedge b$ also in M. (S_n, D) is a sub lattice of (Z_*, D).

4. Define partial ordering on S.

Ans: A relation '$\leq$' on a set S is called a partial ordering on S if it has the following three properties. S is reflexive, anti-symmetric, transitive. A set S together with a partial ordered set or poset.

5. Define Hasse diagram

Ans: Hasse diagram of a finite partially ordered set S is the directed graph whose vertices are the elements of S and there is a directed edge from a to b whenever $a < b$ in S.

6. Simplify the Boolean expression $a'.b.c + a.b'.c + a.b'.c'$ using Boolean algebraic identities.

Ans: $a'.b'.c + a.b'.c + a.b'.c' = a'.b'.c + a.b(c+c')$

$= a'.b'.c + a.b'.(c+c')$

$= a'.b'.c + a.b'.(1)$

$= b'(a'.c + a) = b'.(a+a')(a.c)$

$= ab' + b'.c$

7. Prove that $D_{42} = \{S_{42}, D\}$ is a complemented lattice by finding the complements of all the elements.

Ans:

$D_{42} = \{1, 2, 3, 6, 7, 14, 21, 42\}$.

The complement of 1 is 42, the complement of 2 is 21, the compliment of 3 is 14, the complement of 6 is 7, the compliment of 14 is 3, the complement of 21 is 2, the compliment of 42 is 1, the compliment of 7 is 6.

$\therefore$ Every element has a complement. Hence $D_{42} = \{S_{42}, D\}$ is a complemented lattice.

8. In the poset $(Z^+, /)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Ans: Since 3/9, the integers 3 and 9 are comparable.

For 5, 7 neither 5/7 nor 7/5.

$\therefore$ The integers 5 and 7 are not comparable.

9. When a lattice is called complete?

Ans: A lattice $\langle L, *, \oplus \rangle$ is called complete, if each of its non-empty subsets has a least upper bound and a greatest lower bound.

10. Define direct product of lattice.

Ans: Let $(L, *, \oplus)$ and $(S, \wedge, \vee)$ be two lattices.

The algebraic system $(L \times S, \cdot, +)$ in which the binary operation $+$ and $\cdot$ on $L \times S$ are such that for any (a_1, b_1) and (a_2, b_2) in $L \times S$

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$$

$$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$$

is called the direct product of the lattice $(L, *, \oplus)$ and $(S, \wedge, \vee)$.

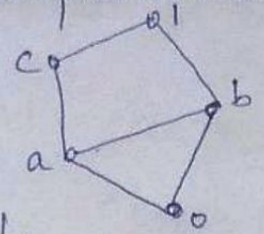
11. Prove that $a + \bar{a}b = a + b$

Ans: $a + \bar{a}b = a + ab + \bar{a}b$ $(\because a = a + ab)$

$= a + b(a + \bar{a}) = a + b$

12. Check the given lattice is complemented lattice or not

Ans:



$b \wedge c = a$ and $b \vee c = 1$

Since $b \wedge a = a$ and $b \vee a = b$.

$\therefore$ b does not have any complement, the given lattice is not complemented lattice.

13. Determine whether the following posets are lattices.

(i) $(\{1, 2, 3, 4, 5\}, /)$ (ii) $(\{1, 2, 4, 8, 16\}, /)$

Ans (i) $(\{1, 2, 3, 4, 5\}, /)$ is not a lattice because there is no upper bound for the pairs $(2, 3)$ and $(3, 5)$.

(ii) $(\{1, 2, 4, 8, 16\}, /)$ is a lattice, since every pair has a LUB and GLB.

14. Reduce the expression $a(a + c)$.

Ans: $a(a+c) = aa + ac = a + ac = a(1 + c) = a$.

15. Prove the involution law $(a')' = a$.

Ans: It is enough to show that $a' + a = 1$ and $a \cdot a' = 0$

By dominance laws of Boolean algebra, we get $a' + a = 1$ and $a \cdot a' = 0$

By commutative laws, we get $a + a' = 1$ and $a' \cdot a = 0$.

$\therefore$ Complement of a' is a (ie) $(a')' = a$.